 Book sections 2.1 - 2.2.4

Chapter 2: Probability (Part 1)

Example 2.1. ***Rolling fair 6-sided dice***

each side is equally likely

(1) *Suppose you roll a fair 6-sided die.*

(a) *What are all the possible outcomes?* ← sample space S

$S = \{1, 2, 3, 4, 5, 6\}$ ← set notation

(b) *What is the probability that you roll a 4?*

$\frac{1}{6} = \frac{\text{one } 4}{\text{size of } S}$ ← 1 possibility

(c) *What is the probability that you roll an even number?*

$\frac{1}{2} = \frac{3}{6}$ → 50% chance, 0.50

(d) *What is the probability that you roll an even number and a 2?* $1/6$

$\{2, 4, 6\} \text{ and } \{2\} = \{2\}$ ($\cap$, intersection)

(e) *What is the probability that you roll a 2 and a 5?* $P(2 \text{ and } 5) = P(\emptyset) = 0$

$\{2\} \text{ and } \{5\} = \emptyset$ ← empty set is a set with no values in it

(f) *What is the probability that you did not roll a 3?*

Direct way: $P(\text{not } 3) = P(\{1,2,4,5,6\}) = \frac{5}{6}$

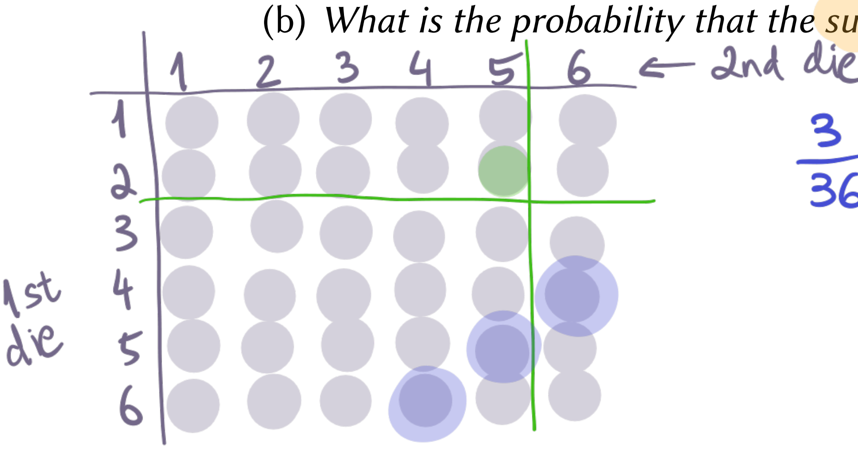
$P(\text{not } 3) = 1 - P(3) = 1 - \frac{1}{6} = \frac{5}{6}$

(2) *Suppose you roll* ***two*** *fair 6-sided dice.*

(a) *What is the probability that you roll a 2 with the first die and a 5 with the second die?*

$\frac{1}{36}$

(b) *What is the probability that the sum of the two dice is 10?*



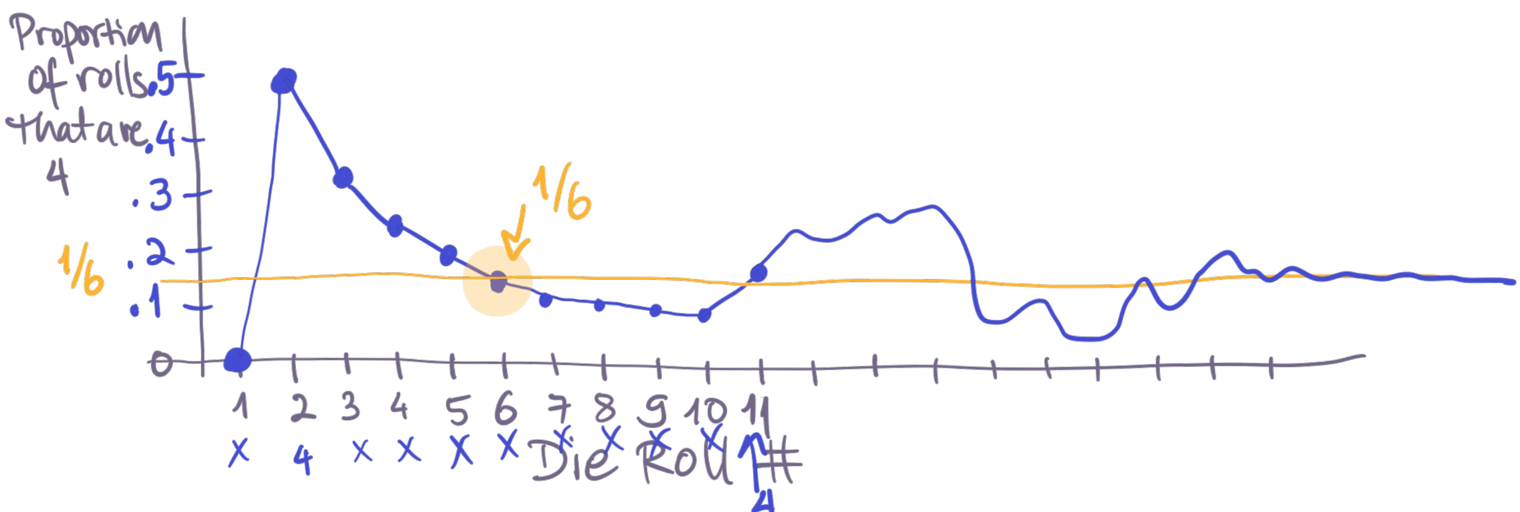
$\frac{3}{36} \left(= \frac{1}{12} \right)$

2.1. What is a probability?

Definition 2.2. *The **probability** of an outcome is the proportion of times the outcome would occur if the random phenomenon could be observed an infinite number of times.*

Law of Large Numbers
As more observations are collected, the proportion of occurrences with a particular outcome converges to the probability p of that outcome.

Estimate P(roll 4)



2.2. Probabilities of Equally Likely Outcomes.

$$P(\text{roll a } 4) = \frac{1}{6} = \frac{\text{one } 4}{6 \text{ possibilities}}$$

$$P(\text{event } A) = \frac{|A|}{|S|} = \frac{\text{size of event } A}{\text{size of sample space}}$$

2.3. Probability Definitions and Rules.

1. The **sample space** S is the set of all possible outcomes.

   $S = \{1, 2, 3, 4, 5, 6\}$ for rolling a 6-sided die

2. **Events** are sets or collections of outcomes.

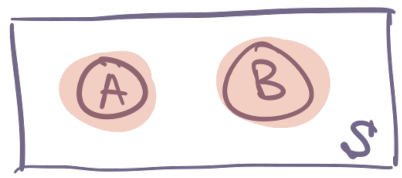
   A = roll an even number $= \{2, 4, 6\}$

3. **Disjoint** or **mutually exclusive** events

   Events A and B are disjoint (or, mutually exclusive) if

   Events can't occur at the same time.

   $P(A \underset{\cap}{\text{and}} B) = 0$ $\qquad A \cap B = \emptyset$

   no overlap!

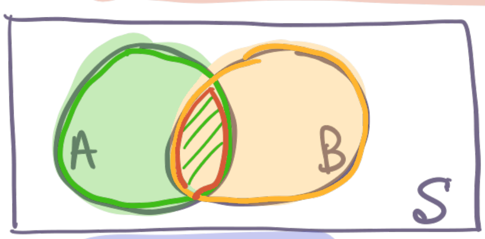
4. **Addition Rule for Disjoint Events**

   If A and B are disjoint events, then the probability that at least one of them will occur is

   $$P(A \text{ OR } B) = P(A) + P(B) - \cancel{P(A \text{ and } B)} \to 0$$

5. **General Addition Rule**

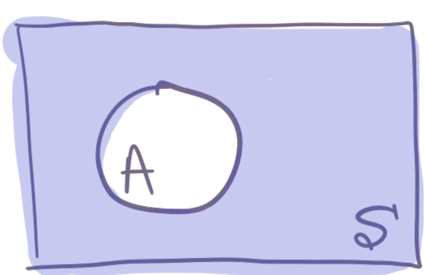
   If A and B are any two events, disjoint or not, then the probability that at least one of them will occur is

   $$P(A \text{ OR } B) = P(A) + P(B) - P(A \underset{\cap}{\text{and}} B)$$

6. **Complement**

   The complement of event A is denoted A^C, and A^C represents all outcomes not in A.

   $$P(A^c) = 1 - P(A)$$

   $$P(A) + P(A^c) = 1$$

   $$P(S) = 1$$

Example 2.3. ***Diabetes and hypertension.*** Class discussion

Diabetes and hypertension are two of the most common diseases in Western, industrialized nations. In the United States, approximately 9% of the population have diabetes, while about 30% of adults have high blood pressure. The two diseases frequently occur together: an estimated 6% of the population have both diabetes and hypertension.

Let D represent the event of having diabetes, and H the event of having hypertension.

(1) *Are the events having hypertension and having diabetes mutually exclusive?*

(2) *Draw a Venn diagram summarizing the variables and their associated probabilities.*

(3) *Calculate the probability of having diabetes or hypertension.*

(4) *What percent of Americans have neither hypertension nor diabetes?*

52 cards, 4 suits: hearts, spades, diamonds, clubs
13 cards in a suit: A, 2, 3, ..., 10, J, Q, K

Example 2.4. *Pulling cards from a standard deck of cards*

(1) *Supposed you randomly pull 5 cards one after another from a standard deck of cards,*

(a) ***replacing** a card before pulling another one. What is the probability of pulling the sequence of* $\diamondsuit, \diamondsuit, \spadesuit, \heartsuit, \spadesuit$*?*

$$P(\diamondsuit \diamondsuit \spadesuit \heartsuit \spadesuit)$$
$$= \frac{13}{52} \cdot \frac{13}{52} \cdot \frac{13}{52} \cdot \frac{13}{52} \cdot \frac{13}{52} = \left(\frac{13}{52}\right)^5 = \left(\frac{1}{4}\right)^5$$

(b) ***without replacing** any of the cards. What is the probability of pulling the sequence of* $\diamondsuit, \diamondsuit, \spadesuit, \heartsuit, \spadesuit$*?*

$$P(\diamondsuit \diamondsuit \spadesuit \heartsuit \spadesuit)$$
$$= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{13}{50} \cdot \frac{13}{49} \cdot \frac{12}{48}$$

$$P(\diamondsuit_1) P(\diamondsuit_2 \mid \diamondsuit_1) P(\spadesuit_3 \mid \diamondsuit_1 \diamondsuit_2) P(\heartsuit_4 \mid \diamondsuit_1 \diamondsuit_2 \spadesuit_3) P(\spadesuit_5 \mid \diamondsuit_1 \diamondsuit_2 \spadesuit_3 \heartsuit_4)$$

(2) *Supposed you pull a red card from the deck. What is the probability that it is a heart?*

$$P(\heartsuit \text{ if card is red}) = \frac{1}{2} = \frac{13 \text{ hearts}}{26 \text{ red}}$$

$$P(\heartsuit \mid \text{red}) = \frac{P(\heartsuit \text{ and red})}{P(\text{red})} = \frac{13/\cancel{52}}{26/\cancel{52}} = \frac{13}{26}$$

given

OR : addition
AND: multiplication

2.4. More Probability Rules.

1. **General Multiplication Rule**
   For events A and B,
   $$\mathbb{P}(A \text{ and } B) = P(A)\,P(B|A)$$
   1st card 2nd card

$$P(A \text{ and } B) = P(B)\,P(A|B)$$

Solve for $P(A|B)$

2. **Conditional Probability Definition**
   The conditional probability of an event A given an event or condition B is
   $$\mathbb{P}(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

3. If A and B are **independent** events, then

   Not the same as mutually exclusive events!
   $P(A \text{ and } B) = 0$

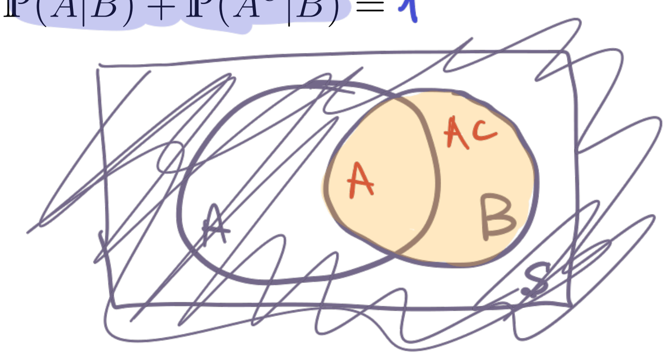
   (a)
   $$\mathbb{P}(A|B) = P(A)$$

   (b) If A & B independent
   $$\mathbb{P}(A \text{ and } B) = P(A)\,P(B)$$

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A)$$

4. **Sum of Conditional Probabilities**
   $\mathbb{P}(A|B)$ is a probability, meaning that it satisfies the usual probability rules. In particular,
   $$\mathbb{P}(A|B) + \mathbb{P}(A^C|B) = 1$$

Recall:
$P(A) + P(A^c) = 1$



Warning!
$P(A|B) + P(A|B^c) \neq 1$

Example 2.5. ***Diabetes and hypertension revisited.***
Diabetes and hypertension are two of the most common diseases in Western, industrialized nations. In the United States, approximately 9% of the population have diabetes, while about 30% of adults have high blood pressure. The two diseases frequently occur together: an estimated 6% of the population have both diabetes and hypertension.
Let D represent the event of having diabetes, and H the event of having hypertension. Is the event of someone being hypertensive independent of the event that someone has diabetes?

Various independence conditions:

$P(D|H) = P(D)$, $P(H|D) = P(H)$, $P(D \text{ and } H) = P(D)P(H)$

Check or verify one of these.

$P(D \text{ and } H) = 0.06$ $\quad P(D) = 0.09$ $\quad P(H) = 0.30$

$\overset{?}{\neq} P(D)P(H) = 0.09(0.30) = 0.027$

D and H are NOT independent

Is $P(D|H) = P(D)$?

$$P(D|H) = \frac{P(D \text{ and } H)}{P(H)} = \frac{0.06}{0.30} = 0.2 \neq 0.09 = P(D)$$

Example 2.6. *Seat belts* Class discussion

Seat belt use is the most effective way to save lives and reduce injuries in motor vehicle crashes. In a 2014 survey, respondents were asked, "How often do you use seat belts when you drive or ride in a car?". The following table shows the distribution of seat belt usage by sex.

		Seat Belt Usage					
		Always	Nearly always	Sometimes	Seldom	Never	Total
Sex	Male	146,018	19,492	7,614	3,145	4,719	180,988
	Female	229,246	16,695	5,549	1,815	2,675	255,980
	Total	375,264	36,187	13,163	4,960	7,394	436,968

(1) *Calculate the marginal probability that a randomly chosen individual always wears seatbelts.*

(2) *What is the probability that a randomly chosen female always wears seatbelts?*

(3) *What is the probability of a randomly chosen individual always wearing seatbelts, given that they are female?*
same as previous

(4) *What is the probability of a randomly chosen individual being female and always wearing seatbelts?*

Seat belts cont'd Class discussion

		Seat Belt Usage					
		Always	Nearly always	Sometimes	Seldom	Never	Total
Sex	Male	146,018	19,492	7,614	3,145	4,719	180,988
	Female	229,246	16,695	5,549	1,815	2,675	255,980
	Total	375,264	36,187	13,163	4,960	7,394	436,968

(5) *What is the probability of a randomly chosen individual always wearing seatbelts, given that they are male?*

(6) *Calculate the probability that an individual who never wears seatbelts is male.*

(7) *Does gender seem independent of seat belt usage?*