Day 15: Simple Linear Regression (Sections 6.1-6.2)

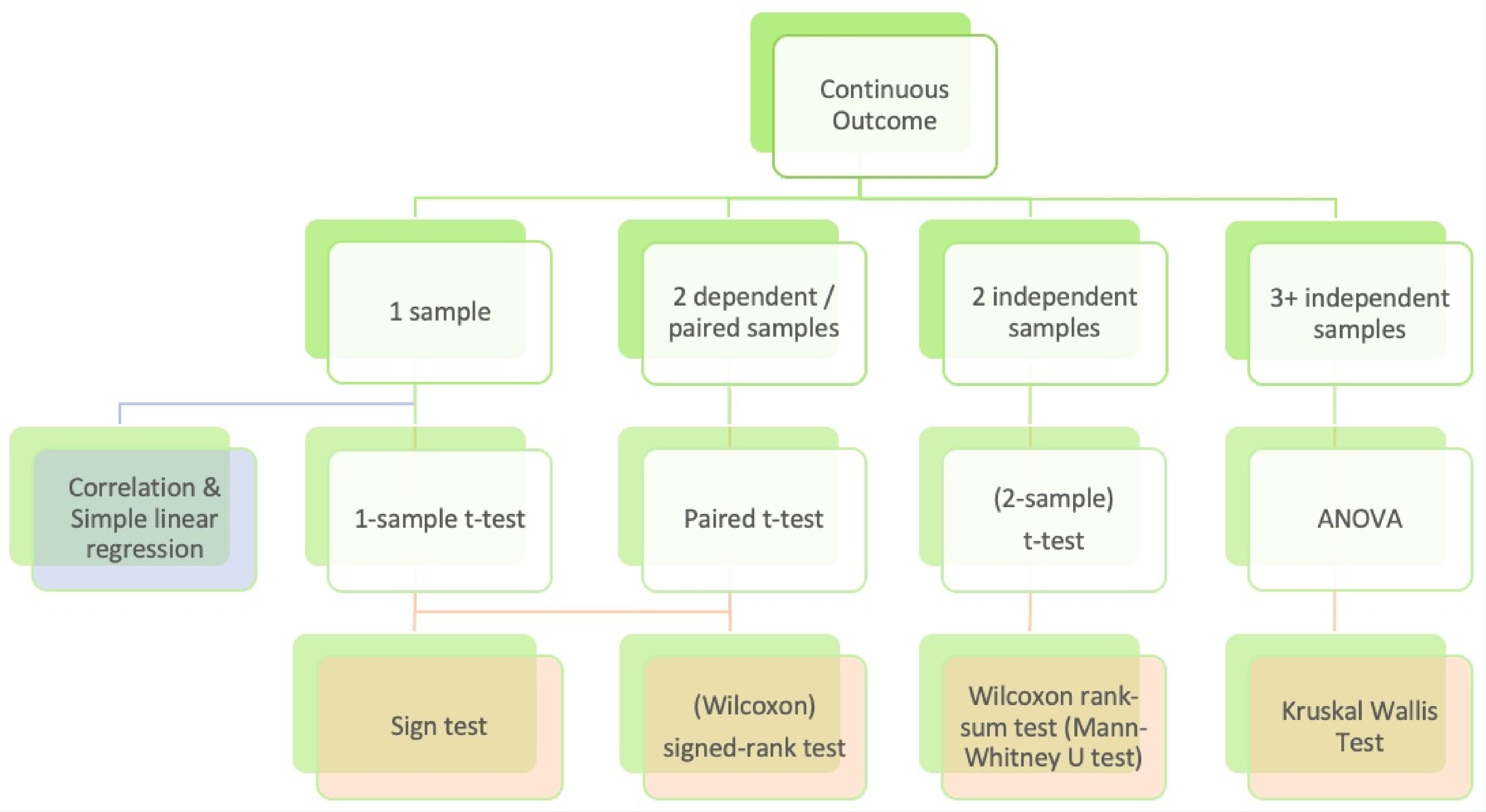
BSTA 511/611

Meike Niederhausen, PhD

OHSU-PSU School of Public Health

~~11/22/2023~~ 11/27/24

Where are we?



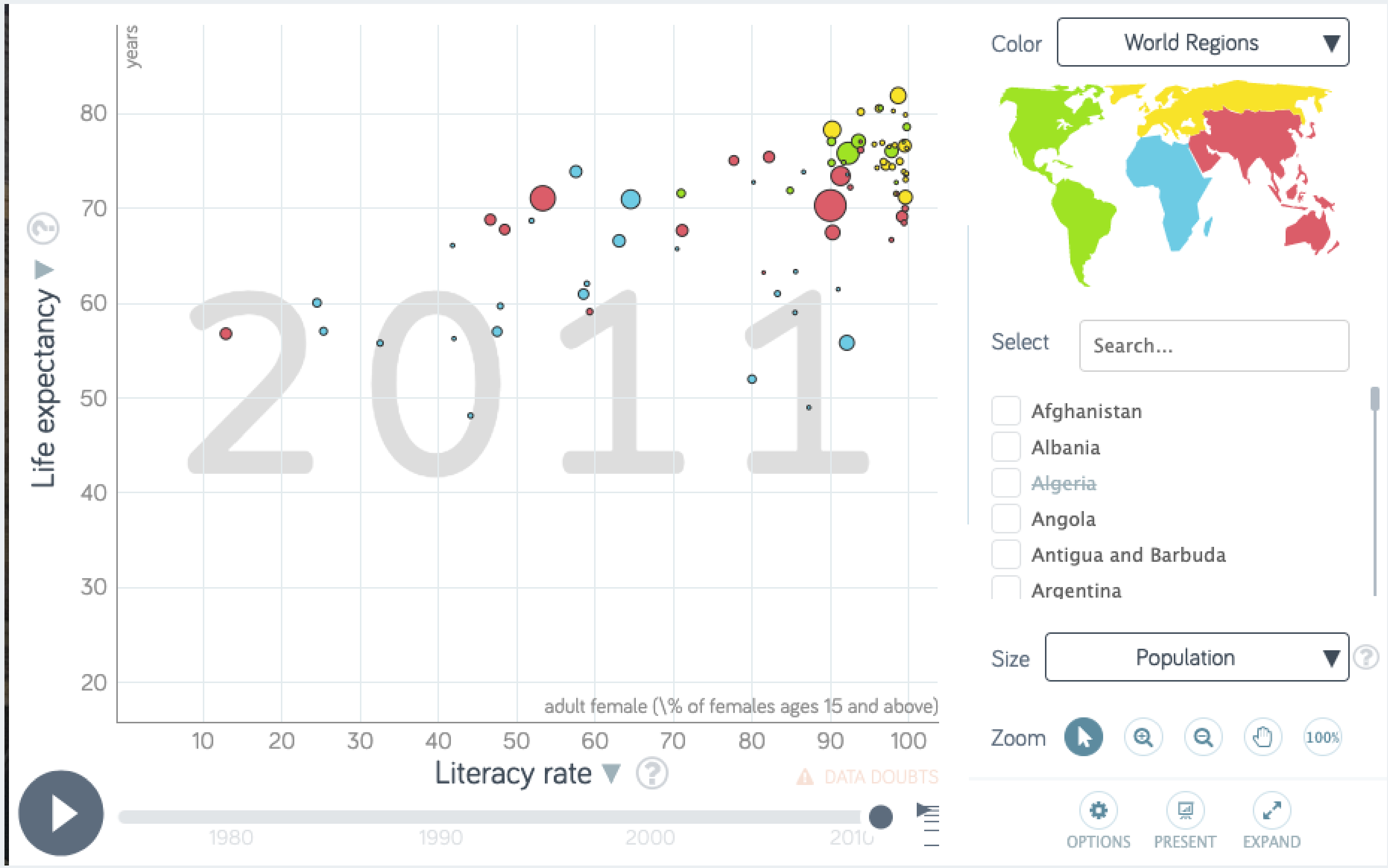
Continuous Outcome
1 sample
2 dependent / paired samples
2 independent samples
3+ independent samples
Correlation & Simple linear regression
1-sample t-test
Paired t-test
(2-sample) t-test
ANOVA
Sign test
(Wilcoxon) signed-rank test
Wilcoxon rank-sum test (Mann-Whitney U test)
Kruskal Wallis Test

Goals for Day 15: Simple Linear Regression (Sections 6.1-6.2)

- Associations between two variables
  - Correlation review
- Simple linear regression: regression with one numerical explanatory variable
- How do we
  - calculate slope & intercept?
  - interpret slope & intercept?
  - Next time:
    - do inference for slope & intercept? (CI, p-value)
    - do prediction with regression line?
- Regression in R
- Residuals
- (Population) regression model
- LINE conditions
  - Does the model fit the data well?
    - Should we be using a line to model the data?
- Should we add additional variables to the model? Take BSTA 512 to answer that (multiple regression)

Life expectancy vs. female adult literacy rate

https://www.gapminder.org/tools/#$model$markers$bubble$encoding$x$data$concep
type=bubbles&url=v1



Dataset description

- Data file: `lifeexp_femlit_water_2011.csv`
- Data were downloaded from https://www.gapminder.org/data/
- 2011 is the most recent year with the most complete data

- **Life expectancy** = the average number of years a newborn child would live if current mortality patterns were to stay the same. Source: https://www.gapminder.org/data/documentation/gd004/

- **Adult literacy rate** is the percentage of people ages 15 and above who can, with understanding, read and write a short, simple statement on their everyday life. Source: http://data.uis.unesco.org/

- **At least basic water source (%)** = the percentage of people using at least basic water services. This indicator encompasses both people using basic water services as well as those using safely managed water services. Basic drinking water services is defined as drinking water from an improved source, provided collection time is not more than 30 minutes for a round trip. Improved water sources include piped water, boreholes or tubewells, protect dug wells, protected springs, and packaged or delivered water.

Get to know the data

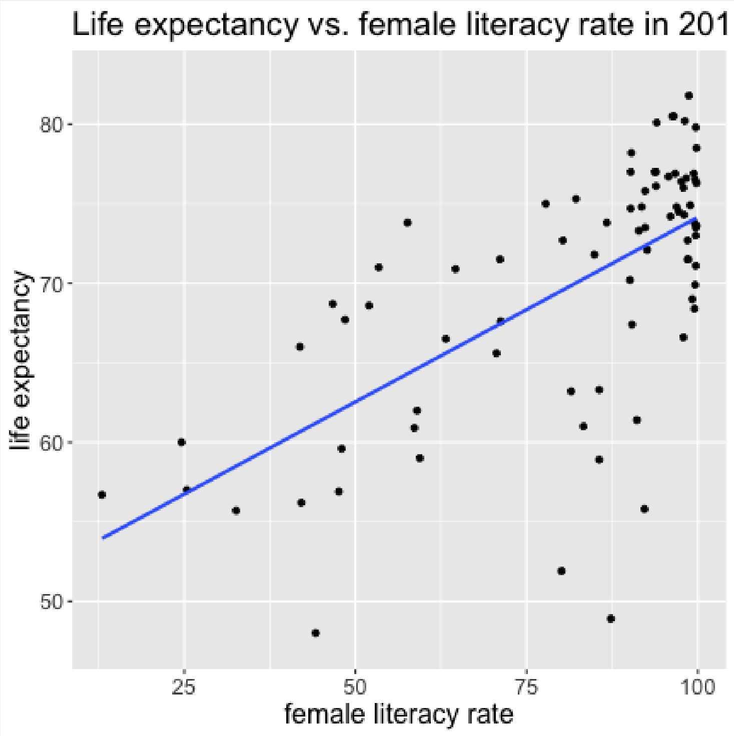
```r
gapm <- read_csv(here::here("data", "lifeexp_femlit_water_2011.csv"))

glimpse(gapm)
```

```
## Rows: 194
## Columns: 5
## $ country                    <chr> "Afghanistan", "Albania", "Algeria", "Andor…
## $ life_expectancy_years_2011 <dbl> 56.7, 76.7, 76.7, 82.6, 60.9, 76.9, 76.0, 7…
## $ female_literacy_rate_2011  <dbl> 13.0, 95.7, NA, NA, 58.6, 99.4, 97.9, 99.5,…
## $ water_basic_source_2011    <dbl> 52.6, 88.1, 92.6, 100.0, 40.3, 97.0, 99.5, …
## $ water_2011_quart           <chr> "Q1", "Q2", "Q2", "Q4", "Q1", "Q3", "Q4", "…
```

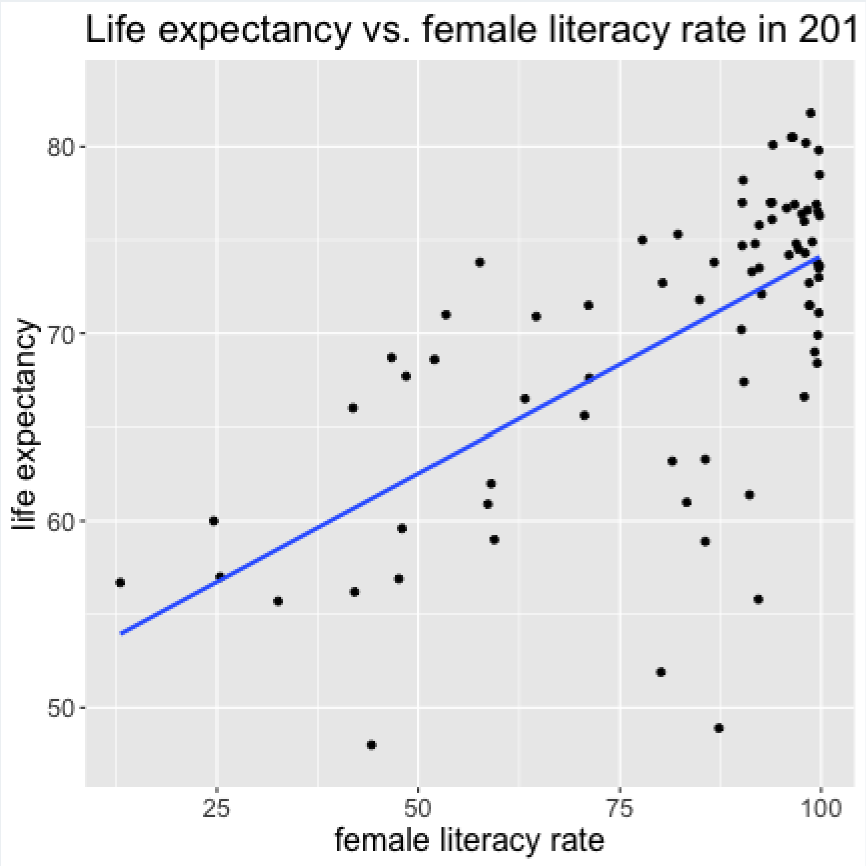
Association between life expectancy and female literacy rate

```
ggplot(gapm,
       aes(x = female_literacy_rate_2011,
           y = life_expectancy_years_2011)) +
  geom_point() +
  labs(x = "female literacy rate",
       y = "life expectancy",
       title = "Life expectancy vs.
             female literacy rate in 2011") +
  geom_smooth(method = "lm",
              se = FALSE)
```



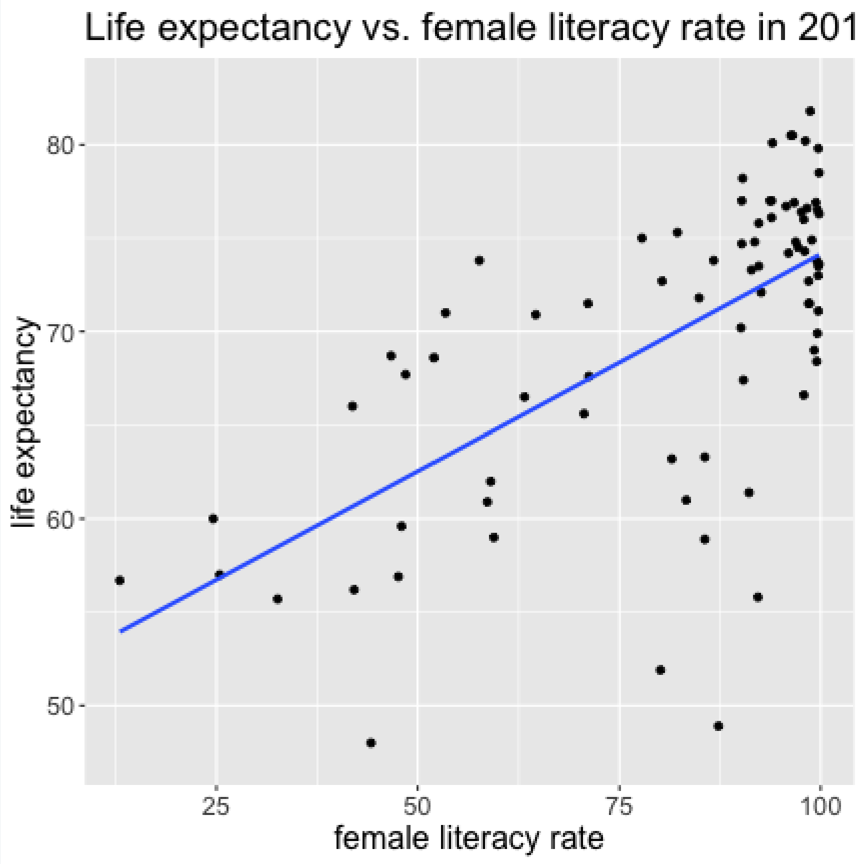
- Is there a relationship between the two variables?
- Is it positive or negative?
- Strong, moderate, or weak?
- Is it linear?

Dependent vs. independent variables



- y = dependent variable (DV)
  - also called the outcome or response variable
- x = independent variable (IV)
  - also called the predictor variable
  - or regressor in a regression analysis
- How to determine which is which?

Correlation between life expectancy and female literacy rate




```
gapm %>% summarize(correlation =
      cor(life_expectancy_years_2011,
          female_literacy_rate_2011,
          use =  "complete.obs")
    )
```

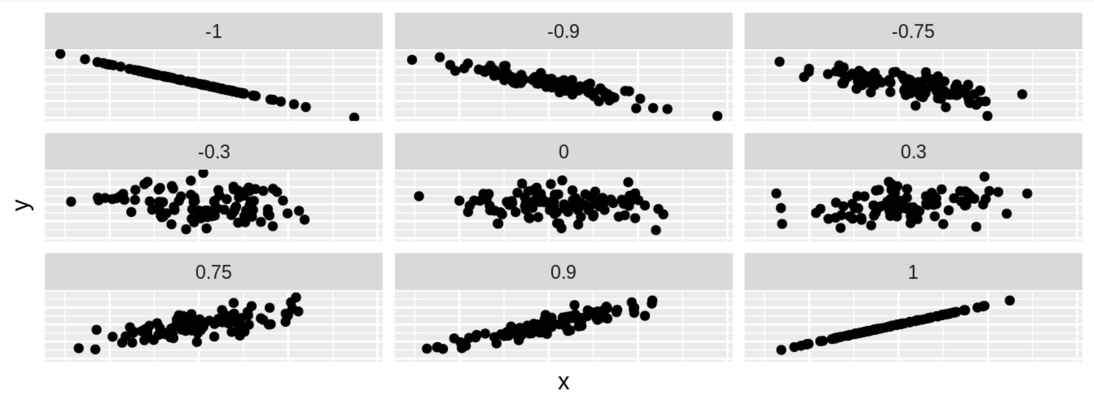
```
## # A tibble: 1 × 1
##   correlation
##         <dbl>
## 1       0.641
```

```
# base R:
cor(gapm$life_expectancy_years_2011,
    gapm$female_literacy_rate_2011,
    use =  "complete.obs")
```

```
## [1] 0.6410434
```

(Pearson) Correlation coefficient (r)

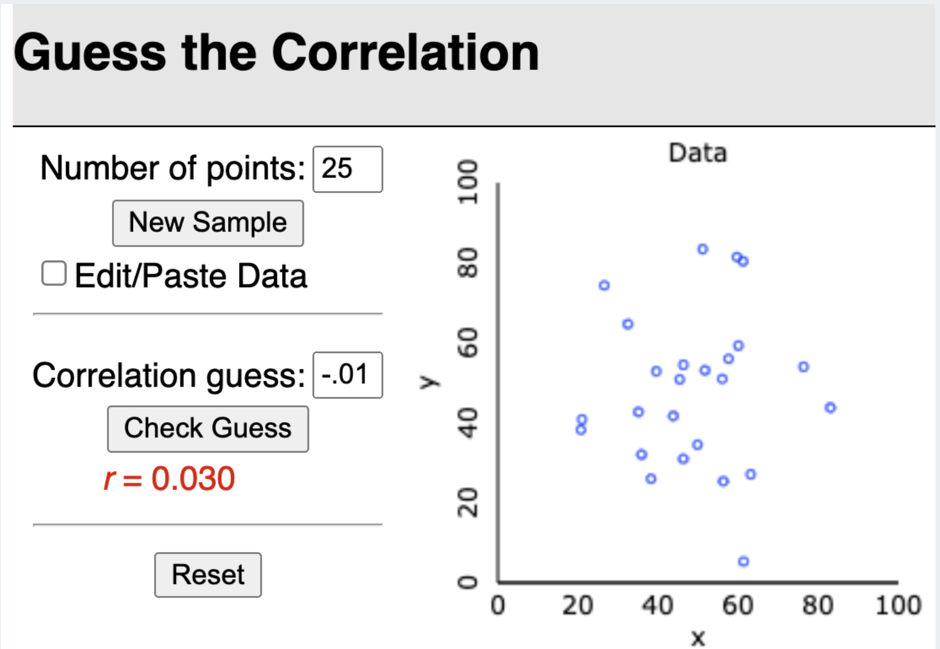
- A *bivariate* summary statistic since calculated using two variables
- **-1** indicates a **perfect negative linear relationship**: As one variable increases, the value of the other variable tends to go down, following a *straight line*.
- **0** indicates **no linear relationship**: The values of both variables go up/down independently of each other.
- **1** indicates a **perfect positive linear relationship**: As the value of one variable goes up, the value of the other variable tends to go up as well in a linear fashion.



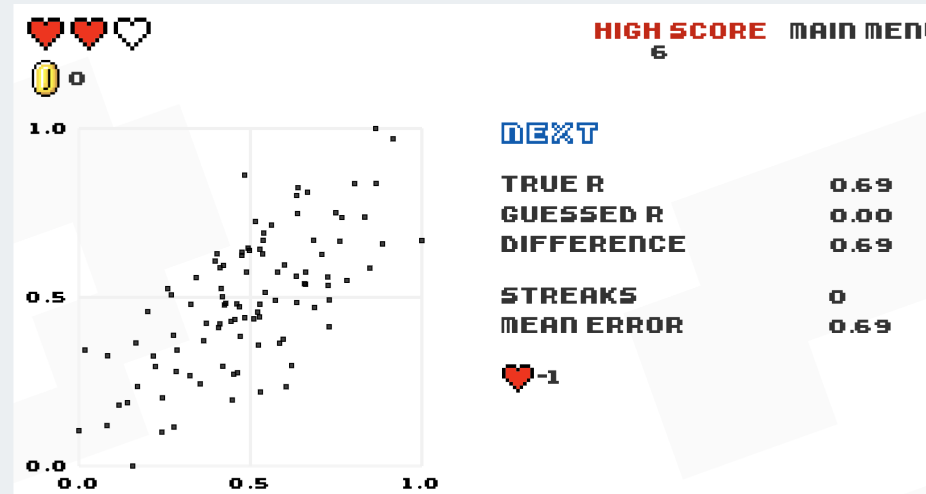
$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

Guess the correlation game!

Rossman & Chance's applet



Or, for the Atari-like experience



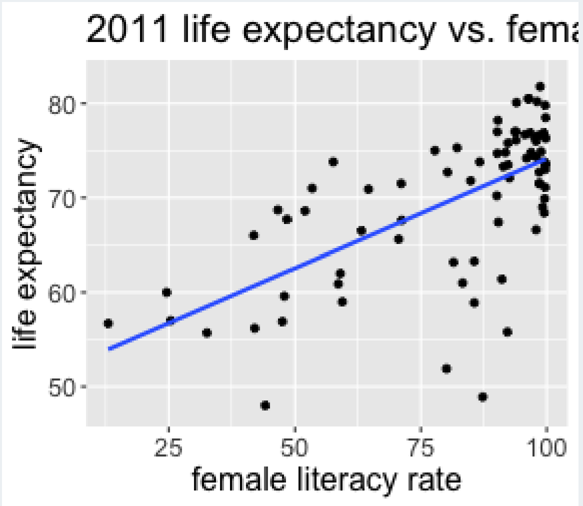
http://guessthecorrelation.com/

Tracks performance of guess vs. actual, error vs. actual, and error vs. trial

https://www.rossmanchance.com/applets/2021/guesscorrelation/GuessCorrelation.html

Regression line = best-fit line

$$\hat{y} = b_0 + b_1 \cdot x$$



- $\hat{y}$ is the predicted outcome for a specific value of x.
- b_0 is the intercept
- b_1 is the slope of the line, i.e., the increase in $\hat{y}$ for every increase of one (unit increase) in x.
  - slope = *rise over run*

- **Intercept**
  - The expected outcome for the y-variable when the x-variable is 0.
- **Slope**
  - For every increase of 1 unit in the x-variable, there is an expected increase of, on average, b_1 units in the y-variable.
  - We only say that there is an expected increase and not necessarily a causal increase.

Correlation coefficient vs. slope

- Not the same!!!
- Directly related though:

$$b_1 = r \frac{s_y}{s_x}$$

where s_x and s_y are the standard deviations of the x and y variables.

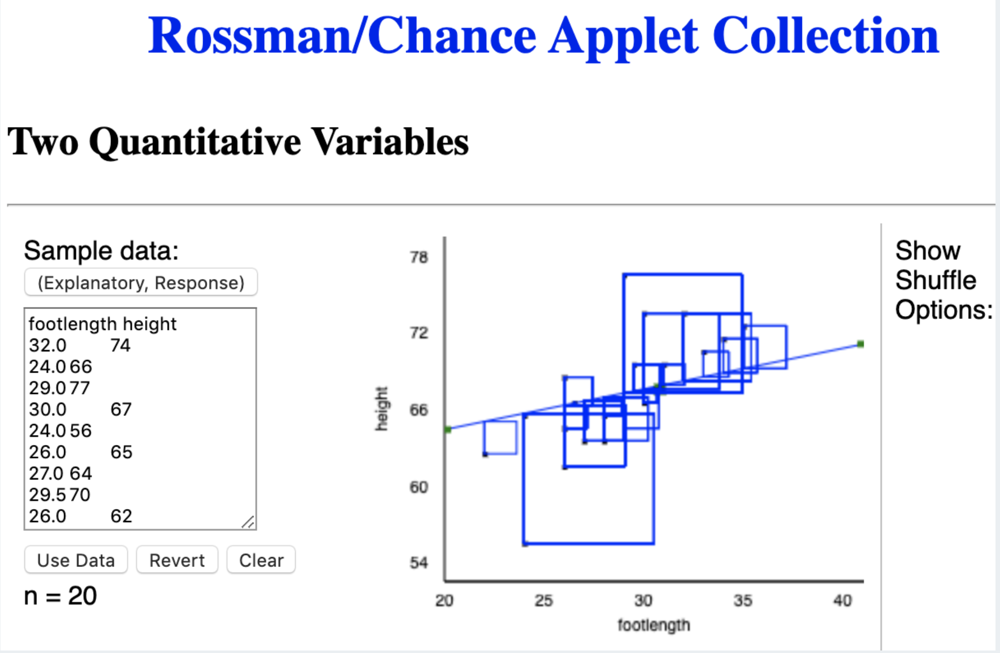
The correlation coefficient can be computed using the formula below.

$$r = \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right)$$

We will be using R to do these calculations.

How is the best-fit line calculated? (1/3)

https://www.rossmanchance.com/applets/2021/regshuffle/regshuffle.htm



- SAE = sum of the absolute values of the residuals (sum of absolute errors)
- SSE = sum of the squared residuals (sum of squared errors)

How is the best-fit line calculated? (2/3)

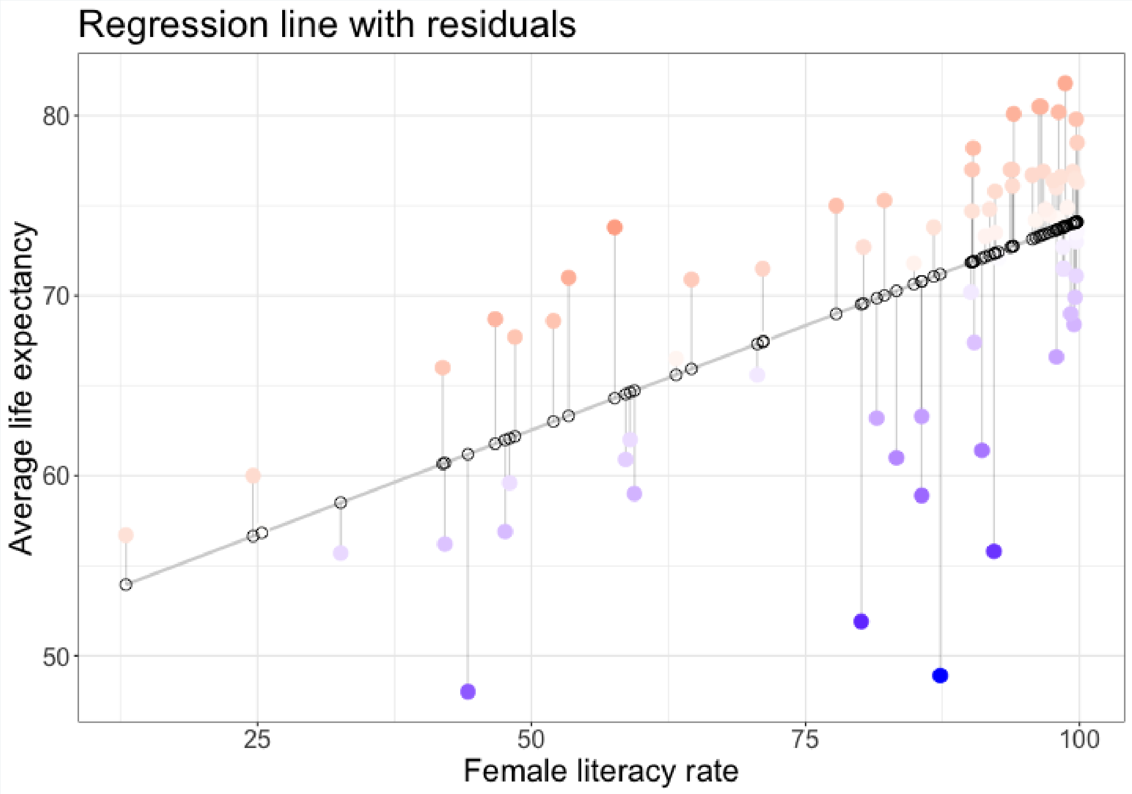
- **Observed values** y_i are the values in the dataset
- **Fitted values** $\hat{y}_i$ are the values that fall on the best-fit line for a specific x_i
- **Residuals** $e_i = y_i - \hat{y}_i$ are the differences between the two.

The best-line is minimizing "how far" the residuals are from the best-fit line

- However, if we add all the residuals we get 0. ($\sum_{i=1}^{n} e_i = 0$)
- Thus instead, we add the squares of the residuals, which is always positive, and minimize the sums of the squares ($\sum_{i=1}^{n} e_i^2 \geq 0$)
  - which is why the best-fit line is often called the **least-squares line**
- This is the same as minimizing the combined area of all the "residual squares" we were looking at in the applet.

Regression model with residuals

- **Observed values** y_i
  - the values in the dataset
- **Fitted values** $\hat{y}_i$
  - the values that fall on the best-fit line for a specific x_i
- **Residuals** $e_i = y_i - \hat{y}_i$
  - the differences between the observed and fitted values



How is the best-fit line calculated? (3/3)

In math-speak, we need to find the values b_0 and b_1 that minimize the equation:

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2$$

where

- y_i are the values in the dataset
- with n datapoints and
- $\widehat{y}_i$ are the fitted values that fall on the best-fit line for a specific value x_i.

How do we find the minimum values of an equation???

If you want to see the mathematical details, check out pages 222-223 (pdf pages 10-11) of https://www.stat.cmu.edu/~hseltman/309/Book/chapter9.pdf.

$$b_0 = \bar{y} - b_1\bar{x}, \qquad b_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = r\frac{s_y}{s_x}$$

Regression in R: summary()

```
model1 <- lm(life_expectancy_years_2011 ~ female_literacy_rate_2011,
                 data = gapm)

summary(model1)
```

```
##
## Call:
## lm(formula = life_expectancy_years_2011 ~ female_literacy_rate_2011,
##     data = gapm)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -22.299  -2.670   1.145   4.114   9.498
##
## Coefficients:
##                           Estimate Std. Error t value Pr(>|t|)
## (Intercept)               50.92790    2.66041  19.143  < 2e-16 ***
## female_literacy_rate_2011  0.23220    0.03148   7.377  1.5e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.142 on 78 degrees of freedom
```

Regression in R: `tidy()` model

```
model1 <- lm(life_expectancy_years_2011 ~ female_literacy_rate_2011,
                 data = gapm)

tidy(model1) %>% gt()
```

term	estimate	std.error	statistic	p.value
(Intercept)	50.9278981	2.66040695	19.142898	3.325312e-31
female_literacy_rate_2011	0.2321951	0.03147744	7.376557	1.501286e-10

Regression equation

The values in the `estimate` column are the intercept and slope of the regression model.

term	estimate	std.error	statistic	p.value
(Intercept)	50.9278981	2.66040695	19.142898	3.325312e-31
female_literacy_rate_2011	0.2321951	0.03147744	7.376557	1.501286e-10

Generic regression equation:

$$\hat{y} = b_0 + b_1 \cdot x$$

Regression equation for our model:

$$\widehat{\text{life expectancy}} = 50.9 + 0.232 \cdot \text{female literacy rate}$$

Interpretation of coefficients

$$\widehat{\text{life expectancy}} = 50.9 + 0.232 \cdot \text{female literacy rate}$$

Interpretation of the intercept for this example. Is it meaningful?

- The average life expectancy in 2011 for countries with no literate female adults was on average 50.9 years.

Interpretation of the slope for this example.

- For every 1 point increase in a country's female literacy rate, we expect an average 0.232 year increase in the country's average life expectancy.

The (Population) Regression Model

The regresison model

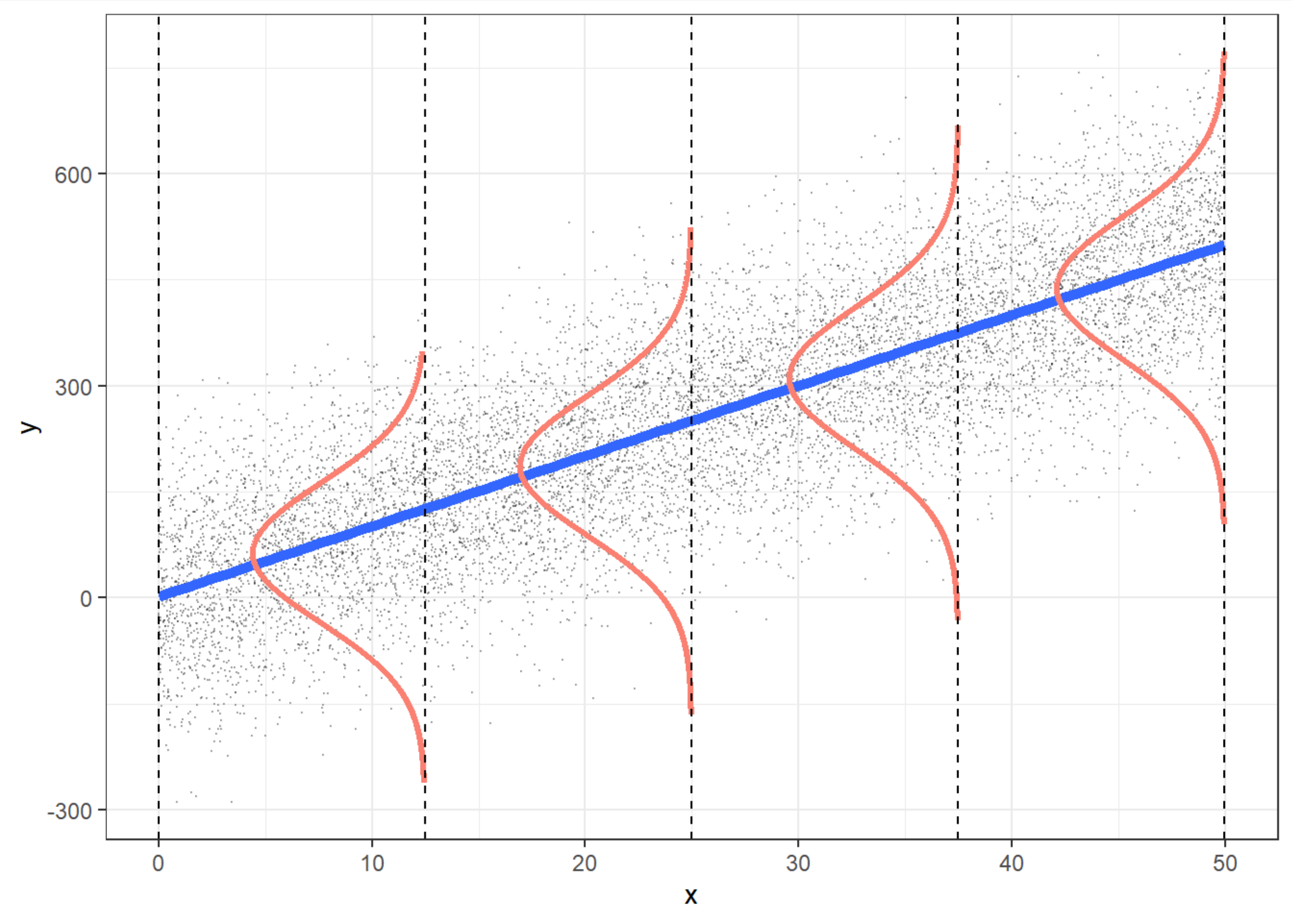
- The (population) regression model is denoted by

$$y = \beta_0 + \beta_1 \cdot x + \epsilon$$

- β_0 and β_1 are unknown population parameters
- ϵ (epsilon) is the error about the line
    - It is assumed to be a random variable with a
        - normal distribution with
        - mean 0 and
        - constant variance σ^2
- The **line** is the average (expected) value of Y given a value of x: $E(Y|x)$.
- The point estimates based on a sample are denoted by $b_0, b_1, s^2_{residuals}$
    - Note: also common notation is $\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\sigma}^2$

The regresison model visually

$$y = \beta_0 + \beta_1 \cdot x + \epsilon, \text{ where } \epsilon \sim N(0, \sigma^2)$$



From https://bookdown.org/roback/bookdown-bysh/ch-MLRreview.html#ordinary-least-squares-ols-assumptions

What are the LINE conditions?

For "good" model fit and to be able to make inferences and predictions based on our models, 4 conditions need to be satisfied.

Briefly:

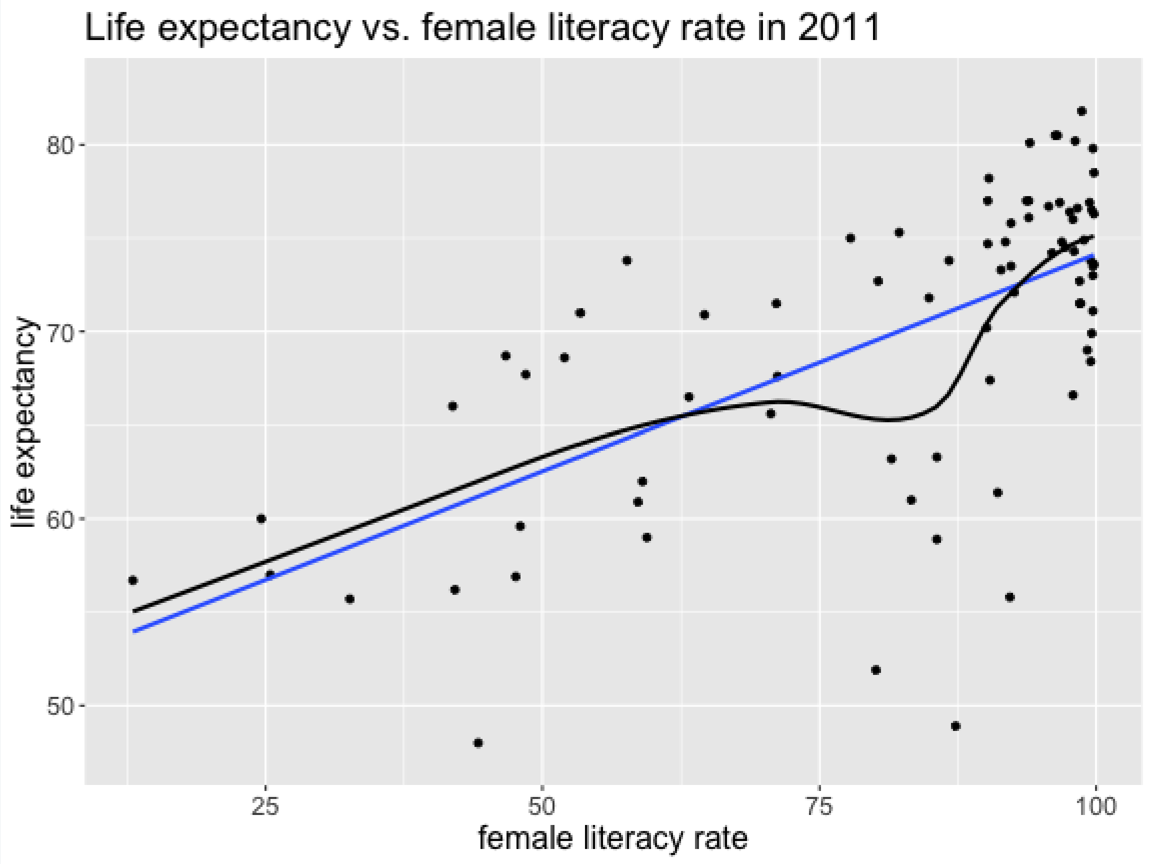
- **L**inearity of relationship between variables
- **I**ndependence of the Y values
- **N**ormality of the residuals
- **E**quality of variance of the residuals (homoscedasticity)

More in depth:

- **L** : there is a linear relationship between the mean response (Y) and the explanatory variable (X),
- **I** : the errors are independent—there's no connection between how far any two points lie from the regression line,
- **N** : the responses are normally distributed at each level of X, and
- **E** : the variance or, equivalently, the standard deviation of the responses is equal for all levels of X.

L: Linearity of relationship between variables

Is the association between the variables linear?



I: Independence of the residuals

- **Are the data points independent of each other?**

- Examples of when they are *not* independent, include

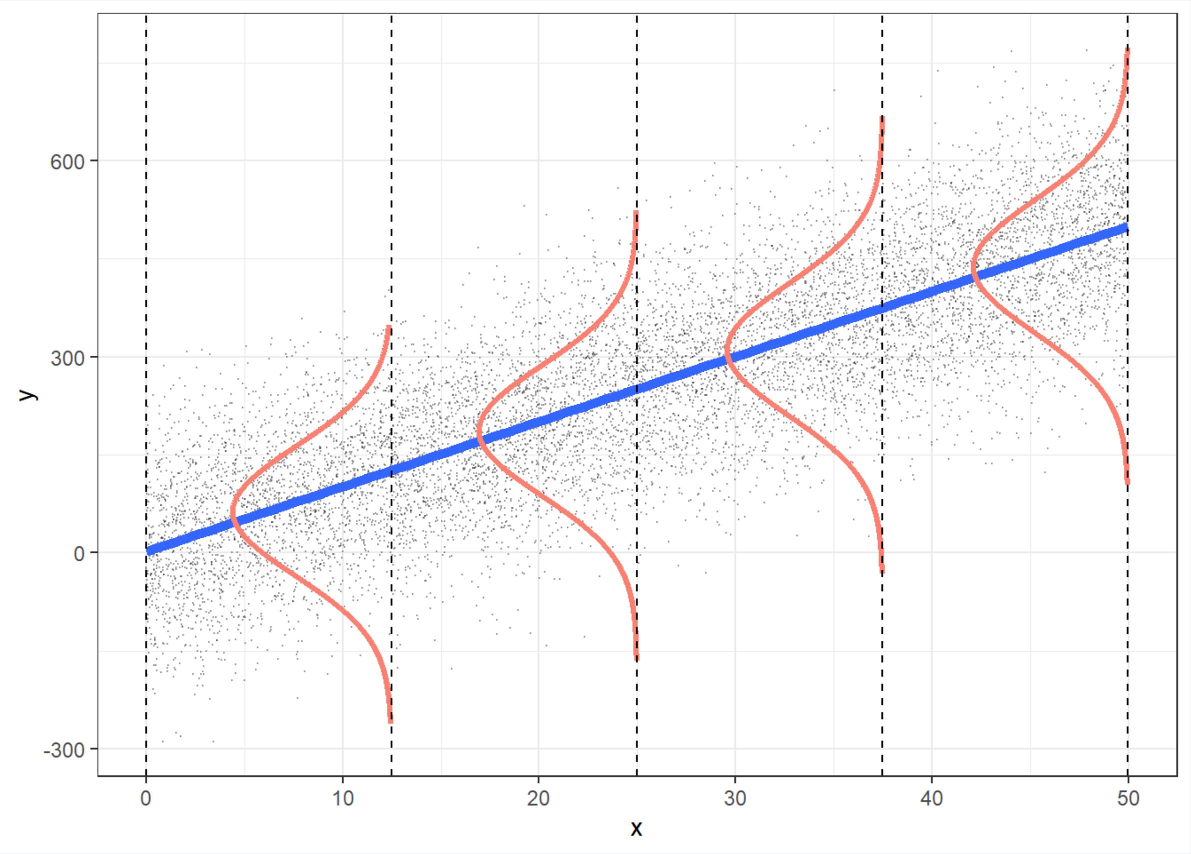
  - repeated measures (such as baseline, 3 months, 6 months)
  - data from clusters, such as different hospitals or families

- This condition is checked by reviewing the study *design* and not by inspecting the data

- How to analyze data using regression models when the Y-values are not independent is covered in BSTA 519 (Longitudinal data)

N: Normality of the residuals

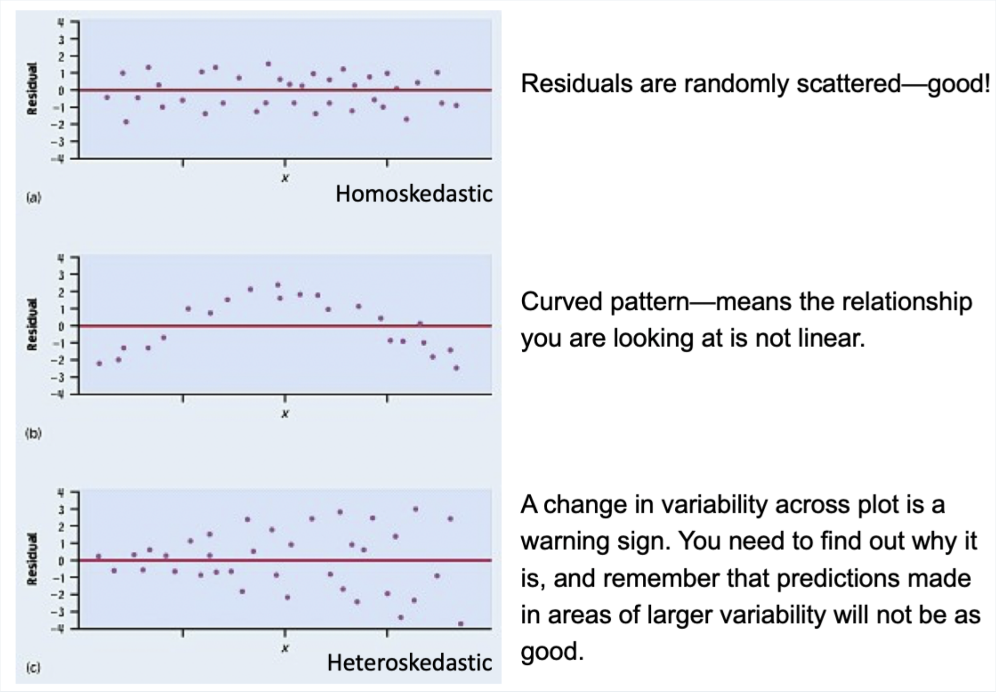
- The responses Y are normally distributed at each level of x
- Next time we will go over how to assess this



From https://bookdown.org/roback/bookdown-bysh/ch-MLRreview.html#ordinary-least-squares-ols-assumptions

E: Equality of variance of the residuals

- The **variance** or, equivalently, the standard deviation of the responses is **equal for all values of x**.
- This is called **homoskedasticity** (top row)
- If there is **heteroskedasticity** (bottom row), then the assumption is not met.

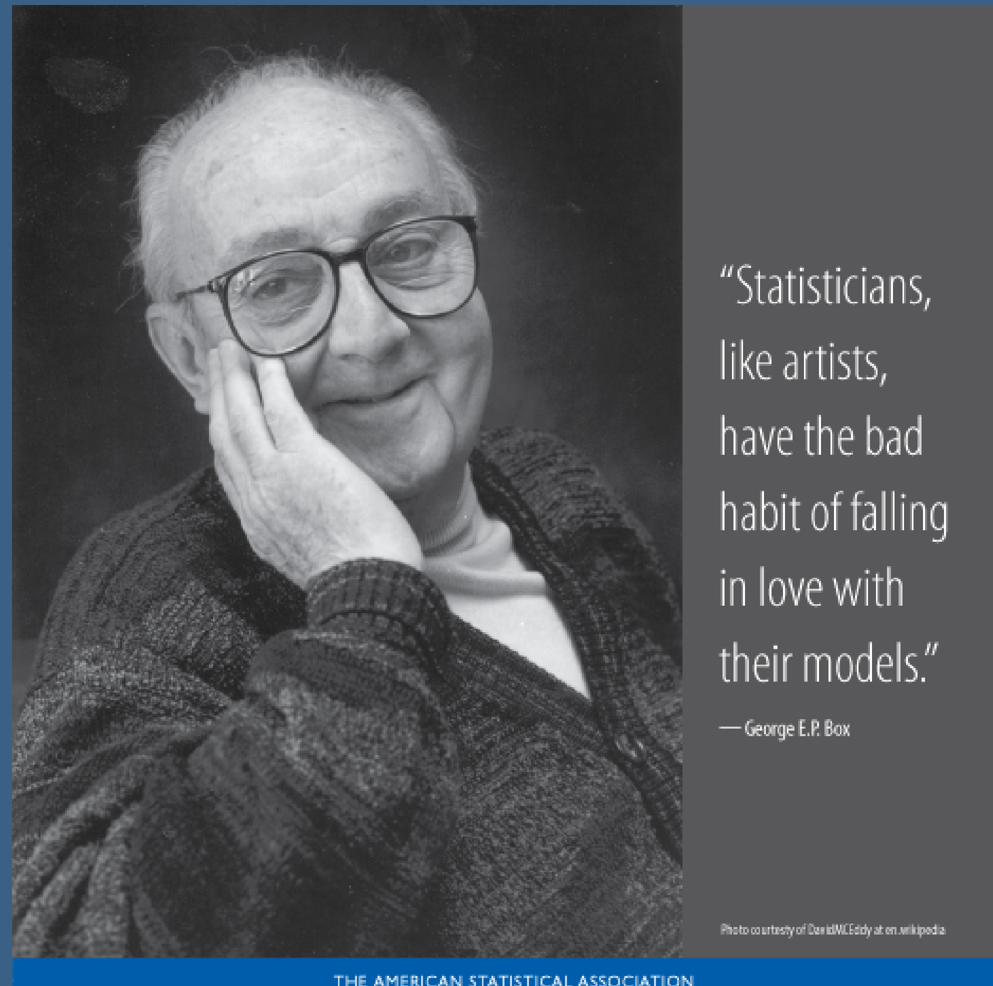


We will discuss how to assess this next time.

All models are wrong, but some are useful

George E.P. Box

(Oct 18, 1919 – March 28, 2013)



https://twitter.com/amstatnews/status/547403146365272064