Section 2.2.5

Day 5 BSTA 511/611

Topics: Sensitivity, specificity, Law of Total Probability, Bayes' Theorem

Chapter 2: Probability (Part 2)

Example 2.7. ***How accurate is rapid testing for COVID-19?***
From the iHealth® website

https://ihealthlabs.com/pages/ihealth-covid-19-antigen-rapid-test-details:

"Based on the results of a clinical study where the iHealth® COVID-19 Antigen Rapid Test was compared to an FDA authorized molecular SARS-CoV-2 test, iHealth® COVID-19 Antigen Rapid Test correctly identified 94.3% of positive specimens and 98.1% of negative specimens."

Suppose you take the iHealth® rapid test.

(1) *What is the probability of a positive test result?*
(2) *What is the probability of having COVID-19 if you get a positive test result?*
(3) *What is the probability of not having COVID-19 if you get a negative test result?*

What information were we given?

First, let's define our events of interest:

- D = *event one has disease (COVID-19)*
- D^c = *event one does not have disease*
- T^+ = *event one tests positive for disease*
- T^- = *event one tests negative for disease* $T^- = (T^+)^c$

Translate given information into mathematical notation:

- *Test correctly gives a positive result 94.3% of the time:*

$P(T^+|D) = 0.943$ sensitivity

- *Test correctly gives a negative result 98.1% of the time:*

$P(T^-|D^c) = 0.981$ specificity

- *What are all the possible scenarios of test results?*

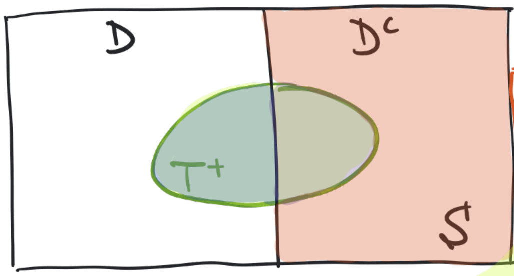
Disease Status

Test Outcome	D	D^c	Total
T^+	$P(T^+\|D)$ sensitivity	$P(T^+\|D^c)$ false positive	$\neq 1$
T^-	$P(T^-\|D)$ false negative	$P(T^-\|D^c)$ specificity	$\neq 1$
Total	1	1	

Given: $\mathbb{P}(T^+|D) = 0.943$, $\mathbb{P}(T^-|D^c) = 0.981$

Solutions to questions

(1) *What is the probability of a positive test result?* $\mathbb{P}(T^+)$



Law of Total Probability

$$\mathbb{P}(T^+) = \mathbb{P}(T^+ \text{ and } D) + \mathbb{P}(T^+ \text{ and } D^c)$$
$$= \mathbb{P}(T^+|D)\mathbb{P}(D) + \mathbb{P}(T^+|D^c)\mathbb{P}(D^c)$$
$$= 0.943\,\mathbb{P}(D) + 0.019\,\mathbb{P}(D^c)$$
$$= 0.943(0.000838) + 0.019(1 - 0.000838)$$
$$= 0.01977431$$

There's approximately a 2% chance of someone in Multnomah County in October 2022 of testing positive for Covid-19.

General Multiplication Rule

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(B|A)\mathbb{P}(A)$$
$$= \mathbb{P}(A|B)\mathbb{P}(B)$$

$$\mathbb{P}(T^+|D^c) + \mathbb{P}(T^-|D^c) = 1$$
$$\mathbb{P}(T^+|D^c) + 0.981 = 1$$
$$\mathbb{P}(T^+|D^c) = 1 - 0.981 = 0.019$$

$\mathbb{P}(D)$ and $\mathbb{P}(D^c)$?

October 2022: 83.8 per 100k in Multnomah County with Covid-19

$\Rightarrow$ Use $\mathbb{P}(D) = \dfrac{83.8}{100{,}000} = 0.000838$

Given: $\mathbb{P}(T^+|D) = 0.943$, $\mathbb{P}(T^-|D^c) = 0.981$

(2) *What is the probability of having COVID-19 if you get a positive test result?*

D T+

Positive Predictive Value (PPV)

$$\mathbb{P}(D|T^+) = \frac{\mathbb{P}(D \text{ and } T^+)}{\mathbb{P}(T^+)}$$

General Multiplication Rule

$$= \frac{\mathbb{P}(T^+|D)\mathbb{P}(D)}{0.01977431}$$

from (1), using the Law of Total Probability

$$= \frac{0.943\,(0.000838)}{0.01977431}$$

Multnomah Co. in October 2022

$$= 0.03996265$$

$\mathbb{P}(D)$	$\mathbb{P}(D\|T^+)$
0.01	0.334
0.10	0.846

Given: $\mathbb{P}(T^+|D) = 0.943$, $\mathbb{P}(T^-|D^c) = 0.981$

(3) *What is the probability of not having* $\overset{D^c}{\text{COVID-19}}$ *if you get a* $\overset{T^-}{\text{negative test}}$ *result?*

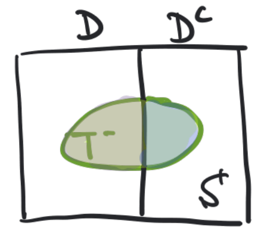
Negative Predictive Value (NPV)

$$\mathbb{P}(D^c|T^-) = \frac{\mathbb{P}(D^c \text{ and } T^-)}{\mathbb{P}(T^-)}$$

$1 - \mathbb{P}(T^+)$

General Multiplication Rule

$$= \frac{\mathbb{P}(T^-|D^c)\mathbb{P}(D^c)}{\underbrace{\mathbb{P}(T^-|D^c)\mathbb{P}(D^c)}_{\mathbb{P}(T^- \text{ and } D^c)} + \underbrace{\mathbb{P}(T^-|D)\mathbb{P}(D)}_{\mathbb{P}(T^- \text{ and } D)}}$$

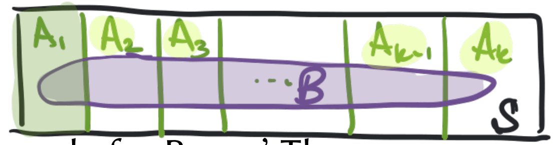


$$= \frac{0.981(1-0.000838)}{0.981(1-0.000838) + (1-0.943)(0.000838)}$$

$$= 0.9999513$$

$\mathbb{P}(D)$	$NPV = \mathbb{P}(D^c|T^-)$
0.01	0.9994134
0.10	0.9935854

Bayes' Theorem (Section 2.2.5)

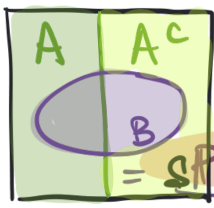


In the previous examples we derived the formula for Bayes' Theorem.

Theorem 2.8 (Bayes' Theorem). *If the sample space S can be split into disjoint events $A_1, A_2, \ldots, A_k$ that make up all possible outcomes in S, and if $\mathbb{P}(A_i) > 0$ for $i = 1, \ldots, k$ and $\mathbb{P}(B) > 0$, then*

$$\mathbb{P}(A_1|B) = \frac{\overbrace{\mathbb{P}(B|A_1) \cdot \mathbb{P}(A_1)}^{\text{General Multiplication Rule}}}{\underbrace{\mathbb{P}(B|A_1) \cdot \mathbb{P}(A_1) + \mathbb{P}(B|A_2) \cdot \mathbb{P}(A_2) + \ldots + \mathbb{P}(B|A_k) \cdot \mathbb{P}(A_k)}_{\text{Law of Total Probability}}}$$

Special case of Bayes' Theorem for sample space being split into A and A^c:



$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^c) \cdot \mathbb{P}(A^c)}$$

$= \mathbb{P}(A \text{ and } B)$

$\mathbb{P}(B)$

Theorem 2.9 (Law of Total Probability). *(denominator of Bayes' Theorem)*

Partition of S

If the sample space S can be split into disjoint events $A_1, A_2, \ldots, A_k$ that make up all possible outcomes in S, and if $\mathbb{P}(A_i) > 0$ for $i = 1, \ldots, k$ and $\mathbb{P}(B) > 0$, then

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \text{ and } A_1) \overset{\text{OR}}{+} \mathbb{P}(B \text{ and } A_2) \overset{\text{OR}}{+} \ldots + \mathbb{P}(B \text{ and } A_k) \\ &= \mathbb{P}(B|A_1) \cdot \mathbb{P}(A_1) + \mathbb{P}(B|A_2) \cdot \mathbb{P}(A_2) + \ldots + \mathbb{P}(B|A_k) \cdot \mathbb{P}(A_k)\end{aligned}$$

Special case of Law of Total Probability for sample space being split into A and A^c:

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \text{ and } A) \overset{\text{OR}}{+} \mathbb{P}(B \text{ and } A^C) \\ &= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^C) \cdot \mathbb{P}(A^C)\end{aligned}$$

Class discussion

Example 2.10. ***Antibody test for COVID-19***

According to the FDA's EUA Authorized Serology Test Performance website, the Abbott AdviseDx SARS-CoV-2 IgG II (Alinity) antibody test for COVID-19 has sensitivity 98.1% and PPV 98.4% when the prevalence is 20%.

Question: What is the specificity of the antibody test?

What information were we given?

First, let's define our events of interest:

- A = *event one has antibodies for COVID-19*
- A^c = *event one does not have antibodies*
- T^+ = *event one tests positive for antibodies*
- T^- = *event one tests negative for antibodies*

Translate given information into mathematical notation:

- *Sensitivity is 98.1%:*
- *PPV is 98.4%:*
- *Prevalence is 20%:*

Solution: